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Testing Closed String Field Theory with Marginal Fields

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Abstract

We study the feasibility of level expansion and test the quartic vertex of closed string field theory by checking the flatness of the potential in marginal directions. The tests, which work out correctly, require the cancellation of two contributions: one from an infinite-level computation with the cubic vertex and the other from a finite-level computation with the quartic vertex. The numerical results suggest that the quartic vertex contributions are comparable or smaller than those of level four fields.

Contents

1 Introduction and summary

Marginal deformations have provided a useful laboratory to deepen our understanding of open string field theory. The effective potential for a marginal field must vanish, but in the level expansion one sees a potential that becomes progressively flatter as the level ℓ is increased [1, 2, 3, 4]. The marginal operator was taken to be $c\partial X$ and corresponds to a constant deformation of the $U(1)$ gauge field in open string theory. The associated spacetime field a_s can be viewed as a Wilson line parameter. For small a_s the potential can be expanded in the form

$$g^2 V_\ell(a_s) = \alpha_4(\ell)\, a_s^4 + \mathcal{O}(a_s^6)\,. \tag{1.1}$$

Numerical evidence was found that the coefficient $\alpha_4(\ell)$ decreases as ℓ increases. Eventually, $\alpha(\ell)$ was elegantly shown to be exactly zero as ℓ goes to infinity [5]. This is, of course, a necessary condition for the potential to vanish completely at infinite level. One can also study large marginal deformations and the relationship between the string field marginal parameter a_s and the conformal field theory marginal parameter [6, 1].

In this paper we use the closed string marginal operator $c\bar{c}\partial X\bar{\partial}X$ to test closed string field theory [7, 8] and to study the feasibility of level expansion in this theory. In order to do this we compute the effective potential for the associated marginal parameter, which we denote as a. We focus on the leading a^4 term in the expansion of this potential for small a. This term receives two contributions. The first one, $\mathcal{C}(\ell)$, arises from the cubic vertex by integration of massive fields of level less than or equal to ℓ. The second contribution, I_4, arises from the elementary quartic vertex of closed string field theory and it has no open string field theory analog. General computations with the quartic vertex are

now possible thanks to the work of Moeller [9]. If we denote by ℓ the maximum level for the massive closed string states that are being integrated, the total potential is

$$\kappa^2\, \mathbb{V}^{tot}_{(\ell)}(a) = \big(\mathcal{C}(\ell) + I_4\ \big)\, a^4 + \mathcal{O}(a^6)\,. \tag{1.2}$$

It is natural to write the coefficient $\mathcal{C}(\ell)$ as

$$\mathcal{C}(\ell) = \sum_{\ell'=0}^{\ell} c(\ell')\,, \tag{1.3}$$

where $c(\ell')$ is the contribution from the massive fields of level ℓ'. Marginality of a requires that the term in parenthesis in (1.2) vanishes as $\ell \to \infty$, or equivalently, that

$$\mathcal{C}(\infty) + I_4 = \sum_{\ell'=0}^{\infty} c(\ell') + I_4\ = 0\,. \tag{1.4}$$

We find strong evidence for this cancellation by computing I_4 and the coefficients $c(\ell)$ to high level. This provides a test of the quartic structure of closed string theory. It is, in fact, the first computation with the quartic vertex in which there is a clear expectation that can be checked. Quartic terms have been computed earlier, most notably the quartic term in the (bulk) tachyon potential [10, 11]. In that case, however, there was no prediction for the magnitude or the sign of the result. Our present work gives us confidence that these early computations are correct.

In open string field theory the level of a cubic interaction is defined to be the sum of the levels of the three states that are coupled. It seems likely that the level of cubic closed string interactions should be defined in the same way. It is less clear how to define a level for quartic interactions in such a way that cubic and quartic contributions may be compared. Equation (1.4) allows us to do such comparison. In particular we can determine the level ℓ_* for which $c(\ell_*) \sim I_4$. Since $|c(\ell)|$ decreases with level, ℓ_* is the level at which inclusion of the quartic interaction seems appropriate. Our results suggest that $\ell_* \gtrsim 4$.

A puzzle arises in the computations. The value of $\mathcal{C}(\infty)$ depends only on the cubic vertex of the string field theory. The value of I_4, which must cancel against $\mathcal{C}(\infty)$, depends on the quartic vertex. It is well known that the quartic vertex is not fully determined by the cubic vertex (although there is a canonical choice). How is it then possible for the cancellation to work for all four-string vertices consistent with the cubic vertex? This happens because of two facts: first, the cubic vertex determines the boundary of the region $\mathcal{V}_{0,4}$ of moduli space that defines the quartic vertex and, second, the integrand for I_4 is a total derivative and the integral reduces to the boundary of $\mathcal{V}_{0,4}$.

Let's review the organization of this paper. In section 2 we state our conventions and carry out the computation of the coefficients $c(\ell)$ for $\ell \le 4$. In section 3 we obtain a simple relation between the coefficients $c(\ell)$ and the analogous coefficients in the open string potential for the marginal Wilson line parameter. Using this relation and the results in [1, 4] we obtain $c(\ell)$ for $\ell \le 20$. With this data we find a fit for $\mathcal{C}(\ell)$ and extrapolate to find $\mathcal{C}(\infty)$. This projected value gives an accurate cancellation

against I_4, the value of which is calculated in section 4. In fact, using the unpublished numerical work of [14, 15] the cancellation works to five significant digits. In section 5, we extend our discussion to the case of the four marginal operators associated with two spacetime directions. The $O(2)$ rotational symmetry implies the existence of two independent structures that can enter into the effective potential to leading (quartic) order in the fields. We compute the contributions to these structures from the cubic and quartic string field vertices and again find convincing cancellations. We offer a discussion of our results in section 6.

2 Marginal field potential from cubic interactions

The bosonic closed string field theory action [7, 8] takes the form

$$S = -\frac{2}{\alpha'}\Big(\frac{1}{2}\langle\Psi|c_0^-\,Q|\Psi\rangle + \frac{\kappa}{3!}\{\Psi, \Psi\,, \Psi\} + \frac{\kappa^2}{4!}\{\Psi,\ \Psi, \Psi, \Psi\} + \cdots\Big). \tag{2.1}$$

Here Q is the BRST operator, $c_0^\pm = \frac{1}{2}(c_0 \pm \bar{c}_0)$, and the string field $|\Psi\rangle$ is a ghost number two state that satisfies $(L_0 - \bar{L}_0)|\Psi\rangle = 0$ and $(b_0 - \bar{b}_0)|\Psi\rangle = 0$. In this paper we only consider states with vanishing momentum. After setting $\alpha' = 2$ and rescaling $\Psi \to \kappa^{-1}\Psi$, the potential $V = -S$ is given by

$$\kappa^2 V = \frac{1}{2}\langle\Psi|c_0^-\,Q|\Psi\rangle + \frac{1}{3!}\{\Psi, \Psi, \Psi\} + \frac{1}{4!}\{\Psi, \Psi, \Psi, \Psi\} + \cdots\ . \tag{2.2}$$

We fix the gauge invariance of the theory using the Siegel gauge $(b_0 + \bar{b}_0)|\Psi\rangle = 0$. The level ℓ of a state is defined as $\ell = L_0 + \bar{L}_0 + 2$. The tachyon state $c_1\bar{c}_1|0\rangle$ has level zero and marginal fields have level two. For a convenient normalization we assume that all spacetime coordinates have been compactified and the volume of spacetime is equal to one. We then use $\langle 0|c_{-1}\bar{c}_{-1}c_0^- c_0^+ c_1\bar{c}_1|0\rangle = 1$, or equivalently,

$$\langle c(z_1)\bar{c}(\bar{w}_1)\, c(z_2)\bar{c}(\bar{w}_2)\, c(z_3)\bar{c}(\bar{w}_3)\rangle = 2(z_1 - z_2)(\bar{w}_1 - \bar{w}_2)(z_1 - z_3)(\bar{w}_1 - \bar{w}_3)(z_2 - z_3)(\bar{w}_2 - \bar{w}_3)\,. \tag{2.3}$$

Since open string field theory uses $\langle c(z_1)c(z_2)c(z_3)\rangle_o = (z_1 - z_2)(z_1 - z_3)(z_2 - z_3)$ we can write

$$\langle c(z_1)c(z_2)c(z_3)\, \bar{c}(\bar{w}_1)\bar{c}(\bar{w}_2)\bar{c}(\bar{w}_3)\rangle = -2\,\langle c(z_1)c(z_2)c(z_3)\rangle_o \,\cdot\, \langle \bar{c}(\bar{w}_1)\bar{c}(\bar{w}_2)\bar{c}(\bar{w}_3)\rangle_o\,. \tag{2.4}$$

This closed/open relation can be used to calculate the cubic coupling of three closed string tachyons:

$$\{c_1\bar{c}_1, c_1\bar{c}_1, c_1\bar{c}_1\} = 2\cdot\langle c_1, c_1, c_1\rangle_o\ \cdot\langle\bar{c}_1, \bar{c}_1, \bar{c}_1\rangle_o = 2\cdot\mathcal{R}^3\cdot\mathcal{R}^3 = 2\mathcal{R}^6\,, \tag{2.5}$$

where $\mathcal{R} \equiv \frac{1}{\rho} = \frac{3\sqrt{3}}{4} \simeq 1.2990$, ρ is the (common) mapping radius of the disks that define the three-string vertex, and $\langle c_1, c_1, c_1\rangle_o = \mathcal{R}^3$ is the coupling of three tachyons in open string field theory (see, for example, [12], eqn. (5.6)).

In this section we only examine quadratic and cubic interactions. We begin by considering the effects of the level zero tachyon t on the potential for the (level two) marginal field a. The string field is therefore

$$|\Psi_0\rangle = t\, c_1\bar{c}_1|0\rangle + a\,\alpha_{-1}\bar{\alpha}_{-1}\, c_1\bar{c}_1|0\rangle\,. \tag{2.6}$$

The subscript on the string field indicates the level of the highest-level *massive* field – in this case zero, because the tachyon is the only massive state. The kinetic term and cubic vertex give the following potential:

$$\kappa^2 V_{(0)} = -t^2 + \frac{1}{3}\mathcal{R}^6\, t^3 + \mathcal{R}^2\, t\, a^2 = -t^2 + \frac{6561}{4096}\, t^3 + \frac{27}{16}\, t a^2. \tag{2.7}$$

To find an effective potential for a we fix values of a, solve for the tachyon field, and substitute back in the potential. For each value of a there are two solutions for the tachyon. One gives the vacuum branch V while the other one gives the marginal branch M. The tachyon values are

$$t^{V/M} = \frac{8192 \pm \sqrt{67108864 - 544195584\, a^2}}{39366}\,. \tag{2.8}$$

As in open string field theory, the marginal parameter is bounded $|a| \leq 0.3512$. It is not clear how higher level and higher order interactions will affect this bound. In the marginal branch we can expand the potential for small a and find $\kappa^2 V_{(0)} \simeq 0.7119\, a^4 + 0.9622\, a^6 + \cdots$. The quartic coefficient can be computed directly using the potential in (2.7) without including the t^3 term. The equation for the tachyon becomes linear and we get

$$\kappa^2 V_{(0)} = \frac{3^6}{2^{10}} a^4 \simeq 0.71191\, a^4 \quad \rightarrow \quad \mathcal{C}(0) = c(0) = 0.71191\,, \tag{2.9}$$

using the notation described in the introduction. In general, to find the contribution to a^4 from a massive field M we only need the kinetic term for M and the coupling $a^2 M$. In terms of Feynman diagrams we are simply computing a tree graph with four external a's, two cubic vertices, and an intermediate massive field.

The string states needed for higher-level computations are built with oscillators $\alpha_{n\leq -1}, \bar\alpha_{n\leq -1}$ of the coordinate X, Virasoro operators $L'_{m\leq -2}, \bar L'_{m\leq -2}$ for the remaining coordinates (thus $c = 25$), and ghost/antighost oscillators. We can list such fields systematically using the generating function:

$$\begin{aligned} f(x,\bar x, y, \bar y) \;\; &= \;\; \prod_{n=1}^{\infty} \frac{1}{1-\alpha_{-n}x^n}\,\frac{1}{1-\bar\alpha_{-n}\bar x^n}\,\prod_{m=2}^{\infty}\frac{1}{1-L'_{-m}x^m}\,\frac{1}{1-\bar L'_{-m}\bar x^m} \\ &\quad \cdot \prod_{\substack{k=-1\\ k\neq 0}}^{\infty} (1+c_{-k}x^k y)(1+\bar c_{-k}\bar x^k \bar y)\prod_{l=2}^{\infty}(1+b_{-l}x^l y^{-1})\,(1+\bar b_{-l}\bar x^l \bar y^{-1})\,. \end{aligned} \tag{2.10}$$

A term of the form $x^n \bar x^{\bar n} y^m \bar y^{\bar m}$ corresponds to a state with $(L_0, \bar L_0) = (n, \bar n)$ and ghost numbers $(G, \bar G) = (m, \bar m)$. A massive field M is relevant to our calculation if the coupling Ma^2 does not vanish. This requires that M have $(G, \bar G) = (1,1)$, an even number of α's, and an even number of $\bar\alpha$'s.

At level two we get three states: the marginal field itself, $c_{-1}c_1|0\rangle$, and $\bar c_{-1}\bar c_1|0\rangle$. One linear combination of the last two is the ghost dilaton and the other is pure gauge. Since none of the three states couples to a^2, we have $c(2) = 0$. At level four $L_0 = \bar L_0 = 1$ and the coefficients of $(x\bar x y \bar y)$ give all possible terms. With the above rule the set is reduced to

$$\begin{aligned} |\Psi_4\rangle \;\; = \;\; & f_1\, c_{-1}\bar c_{-1} + f_2\, L'_{-2}\bar L'_{-2} c_1 \bar c_1 + (f_3\, L'_{-2} c_1 \bar c_{-1} + \tilde f_3\, \bar L'_{-2} c_{-1}\bar c_1) + r_1\, \alpha_{-1}^2 \bar\alpha_{-1}^2 c_1 \bar c_1 \\ & + (r_2\, \alpha_{-1}^2 c_1 \bar c_{-1} + \tilde r_2\, \bar\alpha_{-1}^2 c_{-1}\bar c_1) + (r_3\, \alpha_{-1}^2 \bar L'_{-2} c_1 \bar c_1 + \tilde r_3\, L'_{-2}\bar\alpha_{-1}^2 c_1 \bar c_1)\,. \end{aligned} \tag{2.11}$$

The corresponding terms in the potential are

$$\begin{aligned}\kappa^2 V_{(4)} =& f_1^2 + \frac{121}{432}a^2\, f_1 + \frac{625}{4}f_2^2 + \frac{15625}{1728}a^2\, f_2 - \frac{25}{2}\big[f_3^2+\tilde f_3^2\big] - \frac{1375}{864}a^2\,\big(f_3+\tilde f_3\big)\\ &+ 4r_1^2 + \frac{27}{16}a^2\, r_1 - 2\big[r_2^2+\tilde r_2^2\big] + \frac{11}{16}a^2\,\big(r_2+\tilde r_2\big) + 25\big[r_3^2+\tilde r_3^2\big] - \frac{125}{32}a^2\,\big(r_3+\tilde r_3\big)\,,\end{aligned} \tag{2.12}$$

where we used the conservation laws in [12] to evaluate the cubic interactions. Solving for all the massive fields and substituting back into $V_{(4)}$ we obtain

$$\kappa^2 V_{(4)} = -\frac{19321}{46656}\; a^4 \simeq -0.41412\, a^4 \quad\rightarrow\quad c(4) = -0.41412\,. \tag{2.13}$$

To get the total contribution up to level four we add the above to the result in (2.9):

$$\kappa^2 \mathbb{V}_{(4)} = \frac{222305}{746496}\; a^4 \simeq 0.29780\, a^4 \quad\rightarrow\quad \mathcal{C}(4) = 0.29780\,. \tag{2.14}$$

The contribution from level six string fields vanishes because none of the string fields has even number of α's as well as even number of $\bar\alpha$'s and satisfies the condition that $(G,\bar G)=(1,1)$. Therefore $c(6)=0$. We note that $\mathcal{C}(4)<\mathcal{C}(0)$. To get additional information we turn to open string field theory.

3 Contributions to a^4 calculated using OSFT

As long as we consider closed string states of ghost number $(1,1)$, work in the Siegel gauge, and restrict ourselves to quadratic and cubic interactions, closed string field theory functions as a kind of product of two copies of open string field theory. This will enable us to relate the contributions to the a^4 term in the effective potential to the similar contributions to a_s^4 in the case of open string field theory.

In classical open string field theory the marginal state is $|\phi_a\rangle = \alpha^X_{-1}c_1|0\rangle$ and the marginal field is called a_s [1]. To calculate the quartic potential a_s^4 it suffices to consider

$$g^2\, V_O(\ell) = \sum_{\ell'=0,2,\ldots}^{\ell} -g^2 S_O^{(\ell')}, \qquad -g^2\, S_O^{(\ell)} = \frac{1}{2}\langle\Phi^{(\ell)}|Q_B|\Phi^{(\ell)}\rangle + \langle\Phi^{(\ell)},\phi_a\,,\phi_a\rangle\, a_s^2\,, \tag{3.1}$$

where O is for open string, g is the open string coupling, Q_B is the open string BRST operator, and ℓ is open string level: $(L_0+1)|\Phi^{(\ell)}\rangle = \ell|\Phi^{(\ell)}\rangle$. Only even levels contribute because states of odd level are twist odd and their coupling to a_s^2 vanishes. For each ℓ we sum over all basis states of ghost number one in the Siegel gauge:

$$|\Phi^{(\ell)}\rangle \equiv \sum_i \phi_i^{(\ell)}|\mathcal{O}_i^{(\ell)}\rangle, \qquad L_0\,|\mathcal{O}_i^{(\ell)}\rangle = (\ell-1)\,|\mathcal{O}_i^{(\ell)}\rangle. \tag{3.2}$$

We will leave out the superscript ℓ whenever possible and define

$$m_{ij} \equiv \langle\mathcal{O}_i|c_0|\mathcal{O}_j\rangle, \qquad K_i \equiv \langle\mathcal{O}_i,\phi_a,\phi_a\rangle\,, \tag{3.3}$$

where m_{ij} is a symmetric nondegenerate matrix. In Siegel gauge $Q_B = c_0L_0$ and therefore

$$-\,g^2\, S_O^{(\ell)} = \frac{\ell-1}{2}\;\phi_i\, m_{ij}\;\phi_j + K_i\,\phi_i\, a_s^2\,, \tag{3.4}$$

where summation over repeated i and j indices is implicit. Using matrix notation, $[M]_{ij} = m_{ij}, [K]_i = K_i, [\phi]_i = \phi_i$, we readily find the solution for ϕ and the value of the action:

$$\phi = -\frac{1}{\ell-1}(M^{-1}K)\, a_s^2 \quad \rightarrow \quad -g^2\, S_O^{(\ell)} = -\frac{1}{2(\ell-1)}\, K^T M^{-1} K\, a_s^4\,. \tag{3.5}$$

Back in (3.1) we have

$$g^2 V_O(\ell) = \alpha_4(\ell)\ a_s^4 = a_s^4 \sum_{\ell'=0,2,\ldots}^{\ell} \chi_{\ell'}\,, \quad \text{with} \quad \chi_{\ell'} = -\frac{1}{2(\ell'-1)}\, K^T M^{-1} K\,. \tag{3.6}$$

Let us now turn to closed strings. Because of level matching and the constraints $(b_0 \pm \bar{b}_0)\Psi = 0$, a closed string field of level $L_0 + \bar{L}_0 + 2 = 2\ell$ in the Siegel gauge can be written as a sum of factors: $\Psi^{(2\ell)} = \psi_{ij}\, |\mathcal{O}_i^{(\ell)}\rangle \otimes |\overline{\mathcal{O}}_j^{(\ell)}\rangle$ where the open string states are those in (3.2). Therefore

$$\langle \mathcal{O}_i \otimes \overline{\mathcal{O}}_j | c_0^- Q_B | \mathcal{O}_{i'} \otimes \overline{\mathcal{O}}_{j'}\rangle = \frac{1}{2}\langle \mathcal{O}_i \otimes \overline{\mathcal{O}}_j | c_0 \bar{c}_0 (L_0 + \bar{L}_0) | \mathcal{O}_{i'} \otimes \overline{\mathcal{O}}_{j'}\rangle = 2(\ell-1)\, m_{ii'}\, m_{jj'}, \tag{3.7}$$

where the factor of two in the last step is from the normalization (2.4). For closed strings the marginal state is $G = \alpha_{-1}^X \bar{\alpha}_{-1}^X c_1 \bar{c}_1 |0\rangle$. Since $G = \phi_a \otimes \bar{\phi}_a$, the cubic interaction factorizes: $\{\mathcal{O}_i \otimes \overline{\mathcal{O}}_j, G, G\} = 2\, K_i\, K_j$. Therefore, up to the order a^4, the potential is calculated from

$$\kappa^2\, \mathbb{V}_{(2\ell)} = \sum_{\ell'=0,2,\ldots}^{\ell} -\kappa^2 S^{(2\ell')}\,, \quad -\kappa^2 S^{(2\ell)} = \frac{1}{2}\langle \Psi^{(2\ell)} | c_0^- Q_B | \Psi^{(2\ell)}\rangle + \frac{1}{2}\{\Psi^{(2\ell)}, G, G\}\, a^2\,. \tag{3.8}$$

Our earlier comments allow explicit evaluation:

$$-\kappa^2 S^{(2\ell)} = (\ell-1)\, \psi_{ij}\psi_{i'j'} m_{ii'} m_{jj'} + a^2\, \psi_{ij} K_i K_j. \tag{3.9}$$

The equation of motion for ψ_{ij} is readily solved:

$$\psi_{ij} = -\frac{1}{2(\ell-1)}\, m_{ii'}^{-1} m_{jj'}^{-1}\, K_{i'} K_{j'}\, a^2\,. \tag{3.10}$$

Substituting back into $S^{(2\ell)}$ and using (3.6) we find

$$-\kappa^2 S^{(2\ell)} = -(\ell-1)\Big(-\frac{1}{2(\ell-1)} K^T M^{-1} K\Big)^2\, a^4 = -(\ell-1)\, \chi_\ell^2\, a^4\,. \tag{3.11}$$

We recognize that the contribution to a_s^4 from the open string fields of level ℓ determines the contribution to a^4 from the closed string fields of level 2ℓ. With the notation described in the introduction,

$$\kappa^2 \mathbb{V}_{(2\ell)} = \mathcal{C}(2\ell)\ a^4 = a^4 \sum_{\ell'=0,2,\ldots}^{\ell} c(2\ell')\,, \quad \text{with} \quad c(2\ell) = -(\ell-1)\chi_\ell^2\,. \tag{3.12}$$

The values of $\alpha_4(\ell)$ (recall (3.6)) for $\ell = 0, 2$, and 4 can be read from Table 1 of [1], and values up to $\ell = 10$ from Table 1 of [4] (with extra digits provided by [15]). We reproduce them in Table 1, along

with the corresponding values of χ_ℓ. For $\ell = 0, 2$, we confirm the closed string results of section 2. Since fits in powers of $1/\ell$, where ℓ is open string level, accurately describe the behavior of coefficients in open string effective potentials [13], we use the data for $\ell = 4, 6, 8$, and 10 to fit α_4 to $b_0 + b_1/\ell + b_2/\ell^2$:

$$\alpha_4(\ell) \simeq -0.00026 + \frac{0.35681}{\ell} + \frac{0.12893}{\ell^2}\,. \tag{3.13}$$

This is a good fit since $\alpha_4(\ell)$ must vanish for infinite level. We now use this fit and (3.12) to predict the behavior of $\mathcal{C}(\ell)$ as a function of the closed string level ℓ. It follows from (3.6) and (3.13) that

$$\chi_\ell = \alpha_4(\ell) - \alpha_4(\ell - 2) \simeq -\frac{0.71361}{\ell^2}\,. \tag{3.14}$$

Equation (3.12) then gives

$$\mathcal{C}(2\ell) - \mathcal{C}(2\ell - 4) = -(\ell - 1)\chi_\ell^2 \simeq -\frac{0.50925}{\ell^3}\,. \tag{3.15}$$

This equation is consistent with the extrapolation

$$\mathcal{C}(2\ell) \simeq f_0 + \frac{0.50925}{(2\ell)^2}\,. \tag{3.16}$$

Comparing with the open string result (3.13) we see that the potential converges faster in closed string theory. Given (3.16) we now make a direct fit of $\mathcal{C}$ to $d_0 + d_2/\ell^2 + d_3/\ell^3$ using the closed string data in the table for $\ell = 4, 6, 8$, and 10:

$$\mathcal{C}(2\ell) \simeq 0.25585 + \frac{0.50581}{(2\ell)^2} + \frac{1.06366}{(2\ell)^3}, \tag{3.17}$$

From this projection we find

$$\mathcal{C}(\infty) \simeq 0.25585\,. \tag{3.18}$$

Recalling (1.4), this number must be cancelled by the elementary quartic contribution I_4.

ℓ	χ_ℓ	$\alpha_4(\ell)$	$c(2\ell)$	$\mathcal{C}(2\ell)$
0	0.84375	0.84375	0.71191	0.71191
2	-0.64352	0.20023	-0.41412	0.29780
4	-0.10323	0.09700	-0.03197	0.26583
6	-0.03420	0.06280	-0.00585	0.25998
8	-0.01646	0.04634	-0.00190	0.25808
10	-0.00962	0.03672	-0.00083	0.25725
∞	–	-0.00026	–	0.25585

Table 1: χ_ℓ and $\alpha_4(\ell)$ give the contribution of level ℓ fields and the total contributions up to level ℓ, respectively, to the quartic term in the potential for the Wilson line parameter a_s. The last two columns give the contribution $c(2\ell) = -(\ell-1)\chi_l^2$ of closed string fields of level 2ℓ and the total contributions $\mathcal{C}(2\ell)$ up to level 2ℓ to the quartic term in the potential for the closed string marginal field a. The last row gives the projections from fits.

4 Elementary contribution to a^4

We now compute the coupling of four marginal operators through the four-string elementary vertex of closed string field theory. If all fields have the simple ghost structure $\Psi_i = \mathcal{O}_i c_1 \bar{c}_1 |0\rangle$, with $\mathcal{O}_i$ a primary matter operator of conformal dimension (h_i, h_i), the elementary quartic amplitude is [10]:

$$\{\Psi_1, \Psi_2, \Psi_3, \Psi_4\} = -\frac{2}{\pi} \int_{\mathcal{V}_{0,4}} \frac{dx\, dy\ \langle \mathcal{O}_1(0)\mathcal{O}_2(1)\mathcal{O}_3(\xi)\mathcal{O}_4(t=0)\rangle}{\rho_1^{2-2h_1}\rho_2^{2-2h_2}\rho_3^{2-2h_3}\rho_4^{2-2h_4}}\,. \tag{4.1}$$

Here ρ_i's are the mapping radii and the correlator has the operators inserted at $z = 0, 1, \xi = x + iy$, and $t = 1/z = 0$. In this paper all matter operators have dimension $(1,1)$ and the mapping radii drop out. For the marginal field a the corresponding operator is $G = c\bar{c}\mathcal{O}_{xx}$ with $\mathcal{O}_{xx} = -\partial X \bar{\partial} X$. Using $\langle \partial X(z_1) \partial X(z_2)\rangle = 1/(z_1 - z_2)^2$, as well as the antiholomorphic analog we find

$$\langle \mathcal{O}_{xx}(0)\, \mathcal{O}_{xx}(1)\, \mathcal{O}_{xx}(\xi)\, \mathcal{O}_{xx}(t=0)\rangle = \Big| 1 + \frac{1}{\xi^2} + \frac{1}{(1-\xi)^2}\Big|^2 . \tag{4.2}$$

Therefore, the amplitude $\{G^4\} \equiv \{G, G, G, G\}$ is

$$\{G^4\} = -\,\frac{2}{\pi}\, I_{0,4}\,, \quad \text{with} \quad I_{0,4} \equiv \int_{\mathcal{V}_{0,4}} dxdy \Big| 1 + \frac{1}{\xi^2} + \frac{1}{(1-\xi)^2}\Big|^2 . \tag{4.3}$$

The moduli $\mathcal{V}_{0,4}$ space is comprised of twelve regions, a region $\mathcal{A}$ ([9], Fig. 3) and eleven regions obtained by acting on $\mathcal{A}$ with the transformations $\xi \to 1-\xi$, $\xi \to \frac{1}{\xi}$, $\xi \to \bar{\xi}$, and their compositions [9]. The integrand in (4.3) is invariant under these transformations, so we integrate numerically over $\mathcal{A}$ using the quintic fit provided by Moeller ([9], eqn. (6.5)) and multiply the result by twelve:

$$I_{0,4} = 12 \int_{\mathcal{A}} dxdy \Big| 1 + \frac{1}{\xi^2} + \frac{1}{(1-\xi)^2}\Big|^2 \simeq 9.65029\,. \tag{4.4}$$

The contribution to the potential from the elementary quartic interaction is then

$$\kappa^2 V_4 = \frac{1}{4!}\{G^4\}\, a^4 = -\frac{1}{12\pi}\, I_{0,4}\; a^4 \simeq -\,0.25598\, a^4 \quad \to \quad I_4 = -\,0.25598\,. \tag{4.5}$$

Recalling (3.18), the test indicated in (1.4) gives

$$\mathcal{C}(\infty) + I_4 = 0.25585 - 0.25598 = -0.00013\,. \tag{4.6}$$

The cancellation is impressive: the residue is about 0.05% of I_4.

Best estimates: With the most accurate description of $\mathcal{A}$, Moeller [14] has calculated the integral $I_{0,4}$ and his result gives $I_4 = -0.255\,872(\pm 2)$. Coletti, Sigalov, and Taylor [15] provided us with the χ_ℓ for $\ell \leq 150$. With this data we found $\mathcal{C}(300) = 0.255\,876\,575\,2$, a good estimate of $\mathcal{C}(\infty)$. Fitting $\mathcal{C}$ to $d_0 + d_2/\ell^2 + d_3/\ell^3$ using $2\ell = 204$ to $2\ell = 300$ gives $\mathcal{C}(\infty) = d_0 = 0.255\,870\,873\,1$, which agrees with I_4 to *five* significant digits. With the data for $\ell \leq 78$, M. Beccaria obtained $\mathcal{C}(\infty) = 0.255\,870\,870\,6\,(3)$ using Levin acceleration and the BST algorithm [16].

The cancellation confirms that the sign and the normalization in (4.1) are correct. This is the same sign that implies that the quartic tachyon self-coupling is negative [11, 9]. We have thus extra confidence of the correctness of the early calculation of the quartic term in the tachyon potential.

One can readily see that the integrand in the amplitude $\{G^4\}$ is a total derivative. It is of the form $f(\xi)f(\bar{\xi})d\xi \wedge d\bar{\xi}$, with $f(\xi) = 1 + \frac{1}{\xi^2} + \frac{1}{(1-\xi)^2}$. We then note that $f(\xi) = \partial g(\xi)$ with $g(\xi) = \xi - \frac{1}{\xi} + \frac{1}{(1-\xi)}$, well defined in $\mathcal{V}_{0,4}$ since this region excludes $\xi = 0, 1$, and $\xi = \infty$. Finally, $f(\xi)f(\bar{\xi})d\xi \wedge d\bar{\xi} = \frac{1}{2}\, d\big(g(\xi)f(\bar{\xi})d\bar{\xi} - f(\xi)g(\bar{\xi})d\xi\big)$, which establishes the claim.

5 A moduli space of marginal deformations

If multiple marginal operators define a moduli space the potential for the corresponding fields must vanish identically. An instructive example is provided by the four marginal operators that can be built using the fields X and Y associated with the spacetime coordinates x and y. We will study the potential for the string field

$$|\Psi\rangle = \big(a_{xx}\alpha^X_{-1}\bar{\alpha}^X_{-1} + a_{yy}\alpha^Y_{-1}\bar{\alpha}^Y_{-1} + a_{xy}\alpha^X_{-1}\bar{\alpha}^Y_{-1} + a_{yx}\alpha^Y_{-1}\bar{\alpha}^X_{-1}\big)\, c_1\bar{c}_1|0\rangle\,. \tag{5.1}$$

The marginal fields a_{xx}, a_{yy}, and $a_{xy} + a_{yx}$ are metric deformations while $a_{xy} - a_{yx}$ is a Kalb-Ramond deformation. The marginal fields are conveniently assembled into the two-by-two matrix M:

$$M = \begin{pmatrix} a_{xx} & a_{yx} \\ a_{xy} & a_{yy} \end{pmatrix}. \tag{5.2}$$

It is useful to consider the global $O(2)$ rotational symmetry of the (x, y) plane. The potential for M should be invariant under an $O(2) \times O(2)$ symmetry where the first $O(2)$ rotates the $(\partial X, \partial Y)$ and the second rotates $(\bar{\partial}X, \bar{\partial}Y)$. Consider two rotation matrices R and S ($R^TR = S^TS = 1$). Together they define an element of $O(2) \times O(2)$ which acts on M as $M \to RMS^T$. To quadratic order in M there is an invariant U and a *quasi*-invariant V:

$$U = \text{Tr}(M^TM)\,, \qquad V = \det M\,. \tag{5.3}$$

In general $V \to \pm V$, since R and/or S may have determinant minus one. An example is provided by the parity transformation $S = \text{diag}(1, -1)$. In fact, the Z_2 symmetries that arise because correlators must have even numbers of appearances of holomorphic and antiholomorphic derivatives of each coordinate are taken into account by the various parity transformations. It follows that to quartic order in the fields we have two invariants:

$$U^2 \quad \text{and} \quad V^2\,. \tag{5.4}$$

There are no more independent invariants: the candidate $\text{Tr}(M^TMM^TM)$ is equal to $U^2 - 2V^2$.

The lowest level potential involves the tachyon and M and requires no new computation. Since U contains a^2_{xx} the coefficient coupling t to U is the same as that coupling t to a^2 in (2.7). We thus have, as in (2.9),

$$\kappa^2 V_{(0)} = \frac{3^6}{2^{10}}\, U^2 \simeq 0.7119\, U^2\,. \tag{5.5}$$

At level four 25 states enter the computation. We calculated the effective potential, solved the equations of motion, and verified that all terms assemble into the two anticipated invariants, giving

$$\kappa^2 V_{(4)} = -\frac{19321}{46656}\,U^2 + \frac{344}{729}V^2 \simeq -0.4141\,U^2 + 0.4719\,V^2. \tag{5.6}$$

The total effective potential up to level four from the cubic interactions is therefore:

$$\kappa^2 \mathbb{V}_{(4)} = \frac{222305}{746496}\,U^2 + \frac{344}{729}\,V^2 \simeq 0.2978\,U^2 + 0.4719\,V^2\,. \tag{5.7}$$

At infinite level the coefficients of U^2 and V^2 must be cancelled by elementary quartic interactions. The quartic interactions contribute

$$\kappa^2 V_4 = \gamma_1\,U^2 + \gamma_2 V^2 = \gamma_1 (a_{xx}^4 + 2a_{xx}^2 a_{yy}^2 + \cdots) + \gamma_2 (a_{xx}^2 a_{yy}^2 + \cdots)\,. \tag{5.8}$$

where γ_1 and γ_2 are constants to be determined. The value of γ_1 is determined by our earlier calculation of the quartic amplitude for a. Therefore (4.5) gives $\gamma_1 = -I_{0,4}/(12\pi)$. The coefficient of $a_{xx}^2 a_{yy}^2$ in the potential, to be calculated next, will give us the value of $2\gamma_1 + \gamma_2$, from which we find γ_2.

To compute the elementary quartic amplitude $a_{xx}^2 a_{yy}^2$, we put the operator $\mathcal{O}_{xx}$ associated with a_{xx} at 0 and 1 and the operator $\mathcal{O}_{yy}$ associated with a_{yy} at ξ and ∞. This choice is arbitrary and does not affect the value of the *integrated* correlator; this is not manifest but is guaranteed by the symmetry of the four-string vertex and can be checked explicitly. The matter correlator is:

$$\langle \mathcal{O}_{xx}(0)\mathcal{O}_{xx}(1)\mathcal{O}_{yy}(\xi)\mathcal{O}_{yy}(t=0)\rangle = \Big\langle \partial X\bar\partial X(0)\partial X\bar\partial X(1)\Big\rangle \Big\langle \partial Y\bar\partial Y(\xi)\partial Y\bar\partial Y(t=0)\Big\rangle = 1\,. \tag{5.9}$$

Since the correlator is just one, the amplitude is proportional to the area $A_{0,4}$ of the region $\mathcal{V}_{0,4}$ viewed as a subset of the z plane (with metric $dzd\bar z$):

$$\{\mathcal{O}_{xx}^2\mathcal{O}_{yy}^2\} = -\frac{2}{\pi}\int_{\mathcal{V}_{0,4}} dxdy = -\frac{2}{\pi}\,A_{0,4}\,. \tag{5.10}$$

Since the contribution of a region $\mathcal{S}$ is the same as that of $1-\mathcal{S}$, $\overline{\mathcal{S}}$, and $1-\overline{\mathcal{S}}$ we have

$$A_{0,4} = \int_{\mathcal{V}_{0,4}} dxdy = 4\Big(\int_{\mathcal{A}} + \int_{\frac{1}{\mathcal{A}}} + \int_{\frac{1}{1-\mathcal{A}}}\Big)dxdy = 4\int_{\mathcal{A}} dxdy\Big(1 + \frac{1}{|\xi|^4} + \frac{1}{|1-\xi|^4}\Big) \simeq 6.0774\,. \tag{5.11}$$

Of course, the integrand for area is a total derivative: $d\xi\wedge d\bar\xi = \frac{1}{2}d(\xi d\bar\xi - \bar\xi d\xi)$. Back to the amplitude in question,

$$\kappa^2 V = \frac{6}{4!}\{\mathcal{O}_{xx}^2\mathcal{O}_{yy}^2\}\,a_{xx}^2 a_{yy}^2 = -\frac{1}{2\pi}\,A_{0,4}\,a_{xx}^2 a_{yy}^2\,. \tag{5.12}$$

We thus find:

$$\gamma_2 = -\frac{1}{2\pi}\Big(A_{0,4} - \frac{1}{3}I_{0,4}\Big)\,. \tag{5.13}$$

Collecting our results, equation (5.8) gives

$$\kappa^2 V_4 = -\frac{1}{12\pi}I_{0,4}\,U^2 - \frac{1}{2\pi}\Big(A_{0,4} - \frac{1}{3}I_{0,4}\Big)V^2 \simeq -0.2560\,U^2 - 0.4552\,V^2\,. \tag{5.14}$$

This quartic contribution cancels most of the potential in (5.7). The small residual potential is

$$\kappa^2 V_4^{res} = 0.0418\, U^2 + 0.0167\, V^2\,. \tag{5.15}$$

The data is collected in Table 2. The data for U^2 does not represent a new test, higher level computations would reproduce the result of section 4. The residual coefficient for V^2 is 4% of the original contribution. This is evidence that the infinite-level computation would give the expected cancellation.

level	U^2	V^2
0	0.7119	-
4	-0.4141	0.4719
quartic	-0.2560	-0.4552
residual	0.0418	0.0167

Table 2: Contributions from the given level to the coefficients that multiply the invariants U^2 and V^2 in the effective potential for the marginal fields. The row "quartic" gives the contributions from the elementary quartic interactions. The last row is the residual quartic potential, obtained after adding all contributions.

6 Conclusion

In this paper we have tested the quartic vertex of bosonic closed string field theory and the concrete description of it provided by Moeller [9]. The sign, normalization, and region of integration $\mathcal{V}_{0,4}$ of the quartic interaction were all confirmed. This region comprises the set of four-punctured spheres that are not produced by Feynman graphs built with two cubic vertices and a propagator. Our calculations checked the flatness of the effective potential for marginal parameters; this required the cancellation of cubic contributions of all levels against a finite set of quartic contributions. We examined this cancellation in two examples, one with one marginal direction and one with four marginal directions. In the first one, which we could carry to high level, the cancellation was very accurate and became almost perfect once we used additional numerical data provided by [14, 15]. In the second example, carried to low level, the cancellation was less accurate but still convincing. Amusingly, one of the quartic couplings is equal to the area of $\mathcal{V}_{0,4}$ in the canonical presentation.

The cancellations were guaranteed to happen if closed string field theory reproduces a familiar on-shell fact: the S-matrix element coupling four marginal operators vanishes. Closed string field theory breaks this computation into two pieces, one from Feynman graphs and one from an elementary interaction, thus giving us a consistency test. Our test has verified that the moduli space $\mathcal{M}_{0,4}$ of four punctured spheres is correctly generated by the Feynman graphs and the region $\mathcal{V}_{0,4}$.

We found a simple relation between the quartic terms in the closed string potential for the marginal parameter a and those in the open string potential for the marginal parameter a_s: the contribution to a^4 from closed string fields of level 2ℓ is given by $c(2\ell) = -(\ell-1)\chi_\ell^2$, where χ_ℓ is the contribution to a_s^4 from massive open string fields of level ℓ. Since $\chi_\ell \sim 1/\ell^2$, we have $c(\ell) \sim 1/\ell^3$. Convergence is faster in closed string field theory.

We have gleaned some information about level expansion in closed string field theory by comparing contributions obtained from the cubic and quartic vertices. The natural counter here is the level of the massive fields that are integrated using the cubic vertex and the propagator. Recalling that the quartic vertex contribution is $I_4 \simeq -0.2560$, the column for c in Table 1 shows that $|c(8)| < |I_4| < |c(4)|$, namely, the quartic contribution is smaller than that of level four fields and larger than that of level eight fields. For the case of the invariant V^2 in Table 2, the quartic contribution is only slightly smaller than that from level four fields. These results indicate that the quartic elementary vertex should be included once the level of fields reaches or exceeds four. It remains to be seen if this result holds for other types of computations.

It has been suggested (see [17], for example) that quartic interactions may carry an intrinsic level. The level L_4 of a quartic coupling could be given by $L_4 = \alpha + \beta \sum_{i=1}^{4} \ell_i$, where α and β are constants to be determined. There is scant evidence for any such relation, but we might assume $\beta = 1$ and attempt to estimate α as follows. We learned that $|I_4|$ was bounded by the contributions from level four and level eight massive fields. Since the cubic couplings involve one massive field and two marginal (level two) fields, I_4 is bounded by contributions from level 8 and level 12 interactions. It would be plausible to say that I_4 carries level 10, in which case $\alpha \sim 2$. The same logic applied to the computation of the invariant V^2 would suggest $\alpha \sim 0$. More work will be necessary to uncover a reliable formula for the level of the quartic interaction in closed string field theory.

There are some obvious questions we have not tried to answer. Is the range of a finite or infinite? The cubic tachyon contribution suggests the range is finite, but higher level and higher order interactions could change this result. There are also questions related to the zero-momentum dilaton, a physical, dimension-zero state that fails to satisfy the CFT definition of marginal state because it is not primary. The dilaton theorem, however, implies that the dilaton has a flat potential. This potential is hard to compute because the dilaton is not primary. This computation, which will appear in a separate paper [18], provides new stringent tests of the quartic string vertex, in particular, of the Strebel quadratic differential that determines local coordinates at the punctures. Since the dilaton state exists for general backgrounds its potential is part of the universal structure of string field theory. The dilaton potential is also an important ingredient in any complete computation of the potential for the bulk closed string tachyon.

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